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MATRIX-VALUED HERMITIAN POSITIVSTELLENSATZ, LURKING CONTRACTIONS, AND CONTRACTIVE DETERMINANTAL REPRESENTATIONS OF STABLE POLYNOMIALS

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Dedicated to Leiba Rodman, a dear friend and wonderful colleague, who unfortunately passed away too soon

Abstract. We prove that every matrix-valued rational function F, which is regular on the closure of a bounded domain $\mathcal{D}_{\mathbf{P}}$ in $\mathbb{C}^d$ and which has the associated Agler norm strictly less than 1, admits a finite-dimensional contractive realization

$$F(z) = D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B.$$

Here $\mathcal{D}_{\mathbf{P}}$ is defined by the inequality $\|\mathbf{P}(z)\| < 1$, where $\mathbf{P}(z)$ is a direct sum of matrix polynomials $\mathbf{P}_i(z)$ (so that appropriate Archimedean and approximation conditions are satisfied), and $\mathbf{P}(z)_n = \bigoplus_{i=1}^k \mathbf{P}_i(z) \otimes I_{n_i}$, with some k-tuple n of multiplicities n_i; special cases include the open unit polydisk and the classical Cartan domains. The proof uses a matrix-valued version of a Hermitian Positivstellensatz by Putinar, and a lurking contraction argument. As a consequence, we show that every polynomial with no zeros on the closure of $\mathcal{D}_{\mathbf{P}}$ is a factor of $\det(I - K\mathbf{P}(z)_n)$, with a contractive matrix K.

1. Introduction

It is well-known (see [5, Proposition 11]) that every rational matrix function that is contractive on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ can be realized as

$$F(z) = D + zC(I - zA)^{-1}B, \tag{1.1}$$

with a contractive (in the spectral norm) colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. In several variables, a celebrated result of Agler [1] gives the existence of a realization of the form

$$F(z) = D + CZ_{\mathcal{X}}(I - AZ_{\mathcal{X}})^{-1}B, \qquad Z_{\mathcal{X}} = \bigoplus_{i=1}^{d} z_i I_{\mathcal{X}_i}, \tag{1.2}$$

where $z = (z_1, \ldots, z_d) \in \mathbb{D}^d$ and the colligation $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a Hilbert-space unitary operator (with A acting on the orthogonal direct sum of Hilbert spaces $\mathcal{X}_1, \ldots, \mathcal{X}_d$), for F an operator-valued function analytic on the unit polydisk $\mathbb{D}^d$ whose Agler norm

$$\|F\|_{\mathcal{A}} = \sup_{T \in \mathcal{T}} \|F(T)\| \leq 1.$$

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Here $\mathcal{T}$ is the set of d-tuples $T = (T_1, \ldots, T_d)$ of commuting strict contractions on a Hilbert space. Such functions constitute the Schur–Agler class.

Agler's result was generalized to polynomially defined domains in [3, 6]. Given a d-variable $\ell \times m$ matrix polynomial $\mathbf{P}$, let

$$\mathcal{D}_{\mathbf{P}} = \{z \in \mathbb{C}^d \colon \|\mathbf{P}(z)\| < 1\},$$

and let $\mathcal{T}_{\mathbf{P}}$ be the set of d-tuples T of commuting bounded operators on a Hilbert space satisfying $\|\mathbf{P}(T)\| < 1$. Important special cases are:

(1) When $\ell = m = d$ and $\mathbf{P}(z) = \mathrm{diag}[z_1, \ldots, z_d]$, the domain $\mathcal{D}_{\mathbf{P}}$ is the unit polydisk $\mathbb{D}^d$, and $\mathcal{T}_{\mathbf{P}} = \mathcal{T}$ is the set of d-tuples of commuting strict contractions.

(2) When $d = \ell m$, $z = (z_{rs})$, $r = 1, \ldots, \ell$, $s = 1, \ldots, m$, $\mathbf{P}(z) = [z_{rs}]$, the domain $\mathcal{D}_{\mathbf{P}}$ is a matrix unit ball a.k.a. Cartan's domain of type I. In particular, if $\ell = 1$, then $\mathcal{D}_{\mathbf{P}} = \mathbb{B}^d = \{z \in \mathbb{C}^d \colon \sum_{i=1}^d |z_i|^2 < 1\}$ and $\mathcal{T}_{\mathbf{P}}$ consists of commuting strict row contractions $T = (T_1, \ldots, T_d)$.

(3) When $\ell = m$, $d = m(m+1)/2$, $z = (z_{rs})$, $1 \le r \le s \le m$, $\mathbf{P}(z) = [z_{rs}]$, where for $r > s$ we set $z_{rs} = z_{sr}$, and the domain $\mathcal{D}_{\mathbf{P}}$ is a (complex) symmetric matrix unit ball a.k.a. Cartan's domain of type II.

(4) When $\ell = m$, $d = m(m-1)/2$, $z = (z_{rs})$, $1 \le r < s \le m$, $\mathbf{P}(z) = [z_{rs}]$, where for $r > s$ we set $z_{rs} = -z_{sr}$, and $z_{rr} = 0$ for all $r = 1, \ldots, m$. The domain $\mathcal{D}_{\mathbf{P}}$ is a (complex) skew-symmetric matrix unit ball a.k.a. Cartan's domain of type III.

We notice that Cartan domains of types IV–VI can also be represented as $\mathcal{D}_{\mathbf{P}}$, with a linear $\mathbf{P}$; see [12] and [23].

For $T \in \mathcal{T}_{\mathbf{P}}$, the Taylor joint spectrum $\sigma(T)$ [21] lies in $\mathcal{D}_{\mathbf{P}}$ (see [3, Lemma 1]), and therefore for an operator-valued function F analytic on $\mathcal{D}_{\mathbf{P}}$ one defines $F(T)$ by means of Taylor's functional calculus [22] and

$$\|F\|_{\mathcal{A},\mathbf{P}} := \sup_{T \in \mathcal{T}_{\mathbf{P}}} \|F(T)\|.$$

We say that F belongs to the operator-valued Schur–Agler class associated with $\mathbf{P}$, denoted by $\mathcal{SA}_{\mathbf{P}}(\mathcal{U}, \mathcal{Y})$ if F is analytic on $\mathcal{D}_{\mathbf{P}}$, takes values in the space $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of bounded linear operators from a Hilbert space $\mathcal{U}$ to a Hilbert space $\mathcal{Y}$, and $\|F\|_{\mathcal{A},\mathbf{P}} \le 1$.

The generalization of Agler's theorem mentioned above that has appeared first in [3] for the scalar-valued case and extended in [6] to the operator-valued case, says that a function F belongs to the Schur–Agler class $\mathcal{SA}_{\mathbf{P}}(\mathcal{U}, \mathcal{Y})$ if and only if there exists a Hilbert space $\mathcal{X}$ and a unitary colligation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : (\mathbb{C}^m \otimes \mathcal{X}) \oplus \mathcal{U} \to (\mathbb{C}^\ell \otimes \mathcal{X}) \oplus \mathcal{Y}$$

such that

$$F(z) = D + C(\mathbf{P}(z) \otimes I_{\mathcal{X}})\Big(I - A(\mathbf{P}(z) \otimes I_{\mathcal{X}})\Big)^{-1} B. \tag{1.3}$$

If the Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional, F can be treated as a matrix-valued function (relative to a pair of orthonormal bases for $\mathcal{U}$ and $\mathcal{Y}$). It is natural to ask whether every rational $\alpha \times \beta$ matrix-valued function in the Schur–Agler class $\mathcal{SA}_{\mathbf{P}}(\mathbb{C}^\beta, \mathbb{C}^\alpha)$ has a realization (1.3) with a contractive colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. This question is open, unless when $d = 1$ or F is an inner (i.e., taking unitary boundary values a.e. on the unit torus

$\mathbb{T}^d = \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \colon |z_i| = 1,\ i = 1, \ldots, d\}$) matrix-valued Schur–Agler function on $\mathbb{D}^d$. In the latter case, the colligation matrix can be chosen unitary; see [14] for the scalar-valued case, and [7, Theorem 2.1] for the matrix-valued generalization. We notice here that not every inner function is Schur–Agler; see [9, Example 5.1] for a counterexample.

In the present paper, we show that finite-dimensional contractive realizations of a rational matrix-valued function F exist when F is regular on the closed domain $\overline{\mathcal{D}_{\mathbf{P}}}$ and the Agler norm $\|F\|_{\mathcal{A},\mathbf{P}}$ is strictly less than 1 if $\mathbf{P} = \bigoplus_{i=1}^k \mathbf{P}_i$ and the matrix polynomials $\mathbf{P}_i$ satisfy a certain natural Archimedean condition. The proof has two ingredients: a matrix-valued version of a Hermitian Positivstellensatz [18] (see also [13, Corollary 4.4]), and a lurking contraction argument. For the first ingredient, we introduce the notion of a matrix system of Hermitian quadratic modules and the Archimedean property for them, and use the hereditary functional calculus for evaluations of a Hermitian symmetric matrix polynomial on d-tuples of commuting operators on a Hilbert space. For the second ingredient, we proceed similarly to the lurking isometry argument [1, 8, 3, 6], except that we are constructing a contractive matrix colligation instead of a unitary one.

We then apply this result to obtain a determinantal representation $\det(I - K\mathbf{P}_n)$, where K is a contractive matrix and $\mathbf{P}_n = \bigoplus_{i=1}^k (\mathbf{P}_i \otimes I_{n_i})$, with some k-tuple $n = (n_1, \ldots, n_k)$ of nonnegative integers[1], for a multiple of every polynomial which is strongly stable on $\mathcal{D}_{\mathbf{P}}$. (We recall that a polynomial is called stable with respect to a given domain if it has no zeros in the domain, and strongly stable if it has no zeros in the domain closure.) The question of existence of such a representation for a strongly stable polynomial (without multiplying it with an extra factor) on a general domain $\mathcal{D}_{\mathbf{P}}$ is open.

When $\mathcal{D}_{\mathbf{P}}$ is the open unit polydisk $\mathbb{D}^d$, the representation takes the form $\det(I - KZ_n)$, where $Z_n = \bigoplus_{i=1}^d z_i I_{n_i}$, $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ (see our earlier work [9, 10]). In the cases of $\mathbb{D}$ and $\mathbb{D}^2$, a contractive determinantal representation of a given stable polynomial always exists; see [17, 10]. It also exists in the case of multivariable linear functions that are stable on $\mathbb{D}^d$, $d = 1, 2, \ldots$ [9]. In addition, we showed recently in [11] that in the matrix poly-ball case (a direct sum of Cartan domains of type I) a strongly stable polynomial always has a strictly contractive realization.

The paper is organized as follows. In Section 2, we prove a matrix-valued version of a Hermitian Positivstellensatz. We then use it in Section 3 to establish the existence of contractive finite-dimensional realizations for rational matrix functions from the Schur–Agler class. In Section 4, we study contractive determinantal representations of strongly stable polynomials.

2. Positive Matrix Polynomials

In this section, we extend the result [13, Corollary 4.4] to matrix-valued polynomials. We will write $A \geq 0$ ($A > 0$) when a Hermitian matrix (or a self-adjoint operator on a Hilbert space) A is positive semidefinite (resp., positive definite). For a polynomial with complex matrix coefficients

$$P(w, z) = \sum_{\lambda,\, \mu} P_{\lambda\mu} w^{\lambda} z^{\mu},$$

[1]We use the convention that if $n_i = 0$ then the corresponding direct summand for $\mathbf{P}_n$ is void.

where $w = (w_1, \ldots, w_d)$, $z = (z_1, \ldots, z_d)$, and $w^\lambda = w_1^{\lambda_1} \cdots w_d^{\lambda_d}$, we define

$$P(T^*, T) := \sum_{\lambda,\, \mu} P_{\lambda\mu} \otimes T^{*\lambda} T^\mu,$$

where $T = (T_1, \ldots, T_d)$ is a d-tuple of commuting operators on a Hilbert space. We will prove that P belongs to a certain Hermitian quadratic module determined by matrix polynomials $P_1, \ldots, P_k$ in w and z when the inequalities $P_j(T^*, T) \geq 0$ imply that $P(T^*, T) > 0$.

We denote by $\mathbb{C}[z]$ the algebra of d-variable polynomials with complex coefficients, and by $\mathbb{C}^{\beta\times\gamma}[z]$ the module over $\mathbb{C}[z]$ of d-variable polynomials with the coefficients in $\mathbb{C}^{\beta\times\gamma}$. We denote by $\mathbb{C}^{\gamma\times\gamma}[w, z]_{\mathrm{h}}$ the vector space over $\mathbb{R}$ consisting of polynomials in w and z with coefficients in $\mathbb{C}^{\gamma\times\gamma}$ satisfying $P_{\lambda\mu} = P^*_{\mu\lambda}$, i.e., those whose matrix of coefficients is Hermitian. If we denote by $P^*(w, z)$ a polynomial in w and z with the coefficients $P_{\lambda\mu}$ replaced by their adjoints $P^*_{\lambda\mu}$, then the last property means that $P^*(w, z) = P(z, w)$.

We will say that $\mathcal{M} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ is a matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$ if the following conditions are satisfied:

(1) For every $\gamma \in \mathbb{N}$, $\mathcal{M}_\gamma$ is an additive subsemigroup of $\mathbb{C}^{\gamma\times\gamma}[w, z]_{\mathrm{h}}$, i.e., $\mathcal{M}_\gamma + \mathcal{M}_\gamma \subseteq \mathcal{M}_\gamma$.
(2) $1 \in \mathcal{M}_1$.
(3) For every $\gamma, \gamma' \in \mathbb{N}$, $P \in \mathcal{M}_\gamma$, and $F \in \mathbb{C}^{\gamma\times\gamma'}[z]$, one has $F^*(w)P(w, z)F(z) \in \mathcal{M}_{\gamma'}$.

We notice that $\{\mathbb{C}^{\gamma\times\gamma}[w, z]_{\mathrm{h}}\}_{\gamma\in\mathbb{N}}$ is a trivial example of a matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$.

Remark 2.1. *We first observe that $A \in \mathcal{M}_\gamma$ if $A \in \mathbb{C}^{\gamma\times\gamma}$ is such that $A = A^* \geq 0$. Indeed, using (2) and letting $P = 1 \in \mathcal{M}_1$ and F be a constant row of size γ in (3), we obtain that $0_{\gamma\times\gamma} \in \mathcal{M}_\gamma$ and that every constant positive semidefinite $\gamma \times \gamma$ matrix of rank 1 belongs to $\mathcal{M}_\gamma$, and then use (1). In particular, we obtain that $I_\gamma \in \mathcal{M}_\gamma$. Together with (2) and (3) with $\gamma' = \gamma$, this means that $\mathcal{M}_\gamma$ is a Hermitian quadratic module (see, e.g.,* [20] *for the terminology).*

We also observe that, for each γ, $\mathcal{M}_\gamma$ is a cone, i.e., it is invariant under addition and multiplication with positive scalars.

Finally, we observe that $\mathcal{M}$ respects direct sums, i.e., $\mathcal{M}_\gamma \oplus \mathcal{M}_{\gamma'} \subseteq \mathcal{M}_{\gamma+\gamma'}$. In order to see this we first embed $\mathcal{M}_\gamma$ and $\mathcal{M}_{\gamma'}$ into $\mathcal{M}_{\gamma+\gamma'}$ by using (3) with $P \in \mathcal{M}_\gamma$, $F = [I_\gamma \ 0_{\gamma\times\gamma'}]$ and $P' \in \mathcal{M}_{\gamma'}$, $F' = [0_{\gamma'\times\gamma} \ I_{\gamma'}]$, and then use (1).

The following lemma generalizes [20, Lemma 6.3].

Lemma 2.2. *Let $\mathcal{M}$ be a matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$. The following statements are equivalent:*

(i) *For every $\gamma \in \mathbb{N}$, I_γ is an algebraic interior point of $\mathcal{M}_\gamma$, i.e., $\mathbb{R}I_\gamma + \mathcal{M}_\gamma = \mathbb{C}^{\gamma\times\gamma}[w, z]_{\mathrm{h}}$.*
(ii) *1 is an algebraic interior point of $\mathcal{M}_1$, i.e., $\mathbb{R} + \mathcal{M}_1 = \mathbb{C}[w, z]_{\mathrm{h}}$.*
(iii) *For every $i = 1, \ldots, d$, one has $-w_i z_i \in \mathbb{R} + \mathcal{M}_1$.*

A matrix system $\mathcal{M} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ of Hermitian quadratic modules over $\mathbb{C}[z]$ that satisfies any (and hence all) of properties (i)–(iii) in Lemma 2.2 is called Archimedean. *Proof.* (i)⇒(ii) is trivial.

(ii)⇒(iii) is trivial.

(iii)⇒(i). Let $\mathcal{A}_\gamma = \{F \in \mathbb{C}^{\gamma\times\gamma}[z]\colon -F^*(w)F(z) \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma\}$. It suffices to prove that $\mathcal{A}_\gamma = \mathbb{C}^{\gamma\times\gamma}[z]$ for all $\gamma \in \mathbb{N}$. Indeed, any $P \in \mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ can be written as

$$P(w,z) = \sum_{\lambda,\mu} P_{\lambda\mu} w^\lambda z^\mu = \mathrm{row}_\lambda[w^\lambda I_\gamma][P_{\lambda\mu}]\,\mathrm{col}_\mu[z^\mu I_\gamma] = \\ \mathrm{row}_\lambda[w^\lambda I_\gamma][A_\lambda^* A_\mu - B_\lambda^* B_\mu]\,\mathrm{col}_\mu[z^\mu I_\gamma] = A^*(w)A(z) - B^*(w)B(z),$$

where

$$A(z) = \sum_\mu A_\mu z^\mu \in \mathbb{C}^{\gamma\times\gamma}[z], \quad B(z) = \sum_\mu B_\mu z^\mu \in \mathbb{C}^{\gamma\times\gamma}[z].$$

If $-B^*(w)B(z) \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma$, then so is $P(w,z) = A^*(w)A(z) - B^*(w)B(z)$.

By the assumption, $z_i \in \mathcal{A}_1$ for all $i = 1,\dots,d$. We also have that $\mathbb{C}^{\gamma\times\gamma} \in \mathcal{A}_\gamma$ for every $\gamma \in \mathbb{N}$. Indeed, given $B \in \mathbb{C}^{\gamma\times\gamma}$, we have that $\|B\|^2 I_\gamma - B^*B \geq 0$. By Remark 2.1 we obtain that $\|B\|^2 I_\gamma - B^*B \in \mathcal{M}_\gamma$, therefore $-B^*B \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma$. It follows that $\mathcal{A}_\gamma = \mathbb{C}^{\gamma\times\gamma}[z]$ for all $\gamma \in \mathbb{N}$ if $\mathcal{A}_1$ is a ring over $\mathbb{C}$ and $\mathcal{A}_\gamma$ is a module over $\mathbb{C}[z]$. We first observe from the identity

$$(F^*(w) + G^*(w))(F(z) + G(z)) + (F^*(w) - G^*(w))(F(z) - G(z)) = \\ 2(F^*(w)F(z) + G^*(w)G(z))$$

for $F, G \in \mathcal{A}_\gamma$ that

$$-(F^*(w) + G^*(w))(F(z) + G(z)) = -2(F^*(w)F(z) + G^*(w)G(z)) \\ + (F^*(w) - G^*(w))(F(z) - G(z)) \in \mathbb{R}I_\gamma + \mathcal{M}_\gamma,$$

hence $F + G \in \mathcal{A}_\gamma$. Next, for $F \in \mathcal{A}_\gamma$ and $g \in \mathcal{A}_1$ we can find positive scalars a and b such that $aI_\gamma - F^*(w)F(z) \in \mathcal{M}_\gamma$ and $b - g^*(w)g(z) \in \mathcal{M}_1$. Then we have

$$abI_\gamma - (g^*(w)F^*(w))(g(z)F(z)) = \\ b(aI_\gamma - F^*(w)F(z)) + F^*(w)\Big((b - g(w)^*g(z))I_\gamma\Big)F(z) \in \mathcal{M}_\gamma,$$

Therefore $gF \in \mathcal{A}_\gamma$. Setting $\gamma = 1$, we first conclude that $\mathcal{A}_1$ is a ring over $\mathbb{C}$, thus $\mathcal{A}_1 = \mathbb{C}[z]$. Then, for an arbitrary $\gamma \in \mathbb{N}$, we conclude that $\mathcal{A}_\gamma$ is a module over $\mathbb{C}[z]$, thus $\mathcal{A}_\gamma = \mathbb{C}^{\gamma\times\gamma}[z]$. □

Starting with polynomials $P_j \in \mathbb{C}^{\gamma_j\times\gamma_j}[w,z]_{\mathrm{h}}$, we introduce the sets $\mathcal{M}_\gamma$, $\gamma \in \mathbb{N}$, consisting of polynomials $P \in \mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ for which there exist $H_j \in \mathbb{C}^{\gamma_j n_j\times\gamma}[z]$, for some $n_j \in \mathbb{N}$, $j = 0,\dots,k$, such that

$$P(w,z) = H_0^*(w)H_0(z) + \sum_{j=1}^k H_j^*(w)(P_j(w,z) \otimes I_{n_j})H_j(z). \tag{2.1}$$

Here $\gamma_0 = 1$. We also assume that there exists a constant $c > 0$ such that $c - w_i z_i \in \mathcal{M}_1$ for every $i = 1,\dots,d$. Then $\mathcal{M} = \mathcal{M}_{P_1,\dots,P_k} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ is an Archimedean matrix system of Hermitian quadratic modules generated by $P_1,\dots,P_k$. It follows from Lemma 2.2 that each $\mathcal{M}_\gamma$ is a convex cone in the real vector space $\mathbb{C}^{\gamma\times\gamma}[w,z]_{\mathrm{h}}$ and I_γ is an interior point in the finite topology (where a set is open if and only if its intersection with any finite-dimensional subspace is open; notice that a Hausdorff topology on a finite-dimensional topological vector space is unique).

We can now state the main result of this section.

Theorem 2.3. *Let $P_j \in \mathbb{C}^{\gamma_j \times \gamma_j}[w, z]$, $j = 1, \ldots, k$. Suppose there exists $c > 0$ such that $c^2 - w_i z_i \in \mathcal{M}_1$, for all $i = 1, \ldots, d$. Let $P \in \mathbb{C}^{\gamma \times \gamma}[w, z]$ be such that for every d-tuple $T = (T_1, \ldots, T_d)$ of Hilbert space operators satisfying $P_j(T^*, T) \geq 0$, $j = 1, \ldots, k$, we have that $P(T^*, T) > 0$. Then $P \in \mathcal{M}_\gamma$.*

Proof. Suppose that $P \notin \mathcal{M}_\gamma$. By Lemma 2.2, $I_\gamma \pm \epsilon P \in \mathcal{M}_\gamma$ for $\epsilon > 0$ small enough. By the Minkowski–Eidelheit–Kakutani separation theorem (see, e.g., [15, Section 17]), there exists a linear functional L on $\mathbb{C}^{\gamma \times \gamma}[w, z]_{\mathrm{h}}$ nonnegative on $\mathcal{M}_\gamma$ such that $L(P) \leq 0 < L(I_\gamma)$. For $A \in \mathbb{C}^{1 \times \gamma}[z]$ we define

$$\langle A, A \rangle = L(A^*(w)A(z)).$$

We extend the definition by polarization:

$$\langle A, B \rangle = \frac{1}{4} \sum_{r=0}^{3} i^r \langle A + i^r B, A + i^r B \rangle.$$

We obtain that $(\mathbb{C}^{1 \times \gamma}[z], \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space. Let $\mathcal{H}$ be the Hilbert space completion of the quotient space $\mathbb{C}^{1 \times \gamma}[z]/\{A \colon \langle A, A \rangle = 0\}$. Note that $\mathcal{H}$ is nontrivial since $L(I_\gamma) > 0$.

Next we define multiplication operators M_{z_i}, $i = 1, \ldots, d$, on $\mathcal{H}$. We define M_{z_i} first on the pre-Hilbert space via $M_{z_i}(A(z)) = z_i A(z)$. Suppose that $\langle A, A \rangle = 0$. Since $c^2 - w_i z_i \in \mathcal{M}_1$, it follows that $A^*(w)(c^2 - w_i z_i)A(z) \in \mathcal{M}_\gamma$. Since L is nonnegative on the cone $\mathcal{M}_\gamma$, we have

$$0 \;\leq\; L(A^*(w)(c^2 - w_i z_i)A(z)) \;=\; c^2 \langle A, A \rangle - \langle M_{z_i}(A), M_{z_i}(A) \rangle \;=\; -\langle M_{z_i}(A), M_{z_i}(A) \rangle.$$

Thus, $\langle M_{z_i}(A), M_{z_i}(A) \rangle = 0$, yielding that M_{z_i} can be correctly defined on the quotient space. The same computation as above also shows that $\|M_{z_i}\| \leq c$ on the quotient space, and then by continuity this is true on $\mathcal{H}$. Thus we obtain commuting bounded multiplication operators M_{z_i}, $i = 1, \ldots, d$, on $\mathcal{H}$.

Next, let us show that $P_j(M^*, M) \geq 0$, $j = 1, \ldots, k$, where $M = (M_{z_1}, \ldots, M_{z_d})$. Let $h = [h_r]_{r=1}^{\gamma_j} \in \mathbb{C}^{\gamma_j} \otimes \mathcal{H}$, and moreover assume that h_r are elements of the quotient space $\mathbb{C}^{1 \times \gamma}[z]/\{A \colon \langle A, A \rangle = 0\}$ (which is dense in $\mathcal{H}$). We will denote a representative of the coset h_r in $\mathbb{C}^{1 \times \gamma}[z]$ by $h_r(z)$ with a hope that this will not cause a confusion. Let us compute $\langle P_j(M^*, M)h, h \rangle$. We have

$$P_j(w, z) = \sum_{\lambda,\, \mu} P_{\lambda\mu}^{(j)} w^\lambda z^\mu, \quad P_{\lambda\mu}^{(j)} = [P_{\lambda\mu}^{(j;r,s)}]_{r,s=1}^{\gamma_j}.$$

Then $P_j(M^*, M) = \sum_{\lambda,\,\mu}[P_{\lambda\mu}^{(j;r,s)} M^{*\lambda} M^{\mu}]_{r,s=1}^{\gamma_j}$. Now

$$\Big\langle P_j(M^*, M)h, h\Big\rangle = \sum_{r,\,s=1}^{\gamma_j} \Big\langle \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} M^{*\lambda} M^{\mu} h_r, h_s \Big\rangle =$$

$$\sum_{r,\,s=1}^{\gamma_j} \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} \Big\langle M^{\mu} h_r, M^{\lambda} h_s \Big\rangle =$$

$$\sum_{r,\,s=1}^{\gamma_j} \sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} L\Big(h_s^*(w) w^{\lambda} z^{\mu} h_r(z)\Big) =$$

$$L\Big(\sum_{r,s=1}^{\gamma_j} h_s^*(w)\Big(\sum_{\lambda,\,\mu} P_{\lambda\mu}^{(j;r,s)} w^{\lambda} z^{\mu}\Big) h_r(z)\Big) = L(h^*(w) P_j(w, z) h(z)),$$

which is nonnegative since $h^*(w)P_j(w, z)h(z) \in \mathcal{M}_\gamma$.

By the assumption on P we now have that $P(M^*, M) > 0$. By a calculation similar to the one in the previous paragraph, we obtain that $L(h^*(w)P(w, z)h(z)) > 0$ for all $h \neq 0$. Choose now $h(z) \equiv I_\gamma \in \mathbb{C}^{\gamma\times\gamma}[z]$, and we obtain that $L(P) > 0$. This contradicts the choice of L. □

3. Finite-dimensional Contractive Realizations

In this section, we assume that $\mathbf{P}(z) = \bigoplus_{i=1}^{k} \mathbf{P}_i(z)$, where $\mathbf{P}_i$ are polynomials in $z = (z_1, \ldots, z_d)$ with $\ell_i \times m_i$ complex matrix coefficients, $i = 1, \ldots, k$. Then, clearly, $\mathcal{D}_{\mathbf{P}}$ is a cartesian product of the domains $\mathcal{D}_{\mathbf{P}_i}$. Next, we assume that every d-tuple T of commuting bounded linear operators on a Hilbert space, satisfying $\|\mathbf{P}(T)\| \leq 1$ is a norm limit of elements of $\mathcal{T}_{\mathbf{P}}$. We also assume that the polynomials $P_i(w, z) = I_{m_i} - \mathbf{P}_i^*(w)\mathbf{P}_i(z)$, $i = 1, \ldots, k$, generate an Archimedean matrix system of Hermitian quadratic modules over $\mathbb{C}[z]$. This in particular means that the domain $\mathcal{D}_{\mathbf{P}}$ is bounded, because for some $c > 0$ we have $c^2 - w_i z_i \in \mathcal{M}_1$ which implies that $c^2 - |z_i|^2 \geq 0$, $i = 1, \ldots, d$, when $z \in \mathcal{D}_{\mathbf{P}}$. We notice that in the special cases (1)–(4) in Section 1, the Archimedean condition holds.

We recall that a polynomial convex hull of a compact set $K \subseteq \mathbb{C}^d$ is defined as the set of all points $z \in \mathbb{C}^d$ such that $|p(z)| \leq \max_{w\in K} |p(w)|$ for every polynomial $p \in \mathbb{C}[z]$. A set in $\mathbb{C}^d$ is called polynomially convex if it agrees with its polynomial convex hull.

Lemma 3.1. *$\overline{\mathcal{D}_{\mathbf{P}}}$ is polynomially convex.*

Proof. We first observe that $\overline{\mathcal{D}_{\mathbf{P}}}$ is closed and bounded, hence compact. Next, if $z \in \mathbb{C}^d$ is in the polynomial convex hull of $\overline{\mathcal{D}_{\mathbf{P}}}$, then for all unit vectors $g \in \mathbb{C}^\ell$ and $h \in \mathbb{C}^m$, one has

$$|g^*\mathbf{P}(z)h| \leq \max_{w\in\overline{\mathcal{D}_{\mathbf{P}}}} |g^*\mathbf{P}(w)h| \leq \max_{w\in\overline{\mathcal{D}_{\mathbf{P}}}} \|\mathbf{P}(w)\| \leq 1.$$

Then

$$\|\mathbf{P}(z)\| = \max_{\|g\|=\|h\|=1} |g^*\mathbf{P}(z)h| \leq 1,$$

therefore $z \in \overline{\mathcal{D}_{\mathbf{P}}}$. □

Lemma 3.2. *There exists a d-tuple $T_{\max}$ of commuting bounded linear operators on a separable Hilbert space satisfying $\|\mathbf{P}(T_{\max})\| \leq 1$ and such that*

$$\|q(T_{\max})\| = \|q\|_{\mathcal{A},\mathbf{P}}$$

for every polynomial $q \in \mathbb{C}[z]$.

Proof. The proof is exactly the same as the one suggested in [19, Page 65 and Exercise 5.6] for the special case of commuting contractions and the Agler norm $\|\cdot\|_{\mathcal{A}}$ associated with the unit polydisk, see the first paragraph of Section 1. Notice that the boundedness of $\mathcal{D}_{\mathbf{P}}$ guarantees that $\|q\|_{\mathcal{A},\mathbf{P}} < \infty$ for every polynomial q. □

Lemma 3.3. *Let F be an $\alpha \times \beta$ matrix-valued function analytic on $\overline{\mathcal{D}_{\mathbf{P}}}$. Then $\|F\|_{\mathcal{A},\mathbf{P}} < \infty$.*

Proof. Since F is analytic on some open neighborhood of the set $\overline{\mathcal{D}_{\mathbf{P}}}$ which by Lemma 3.1 is polynomially convex, by the Oka–Weil theorem (see, e.g., [2, Theorem 7.3]) for each scalar-valued function F_{ij} there exists a sequence of polynomials $Q_{ij}^{(n)} \in \mathbb{C}^{\alpha\times\beta}[z]$, $n \in \mathbb{N}$, which converges to F_{ij} uniformly on $\overline{\mathcal{D}_{\mathbf{P}}}$. Therefore the sequence of matrix polynomials $Q^{(n)} = [Q_{ij}^{(n)}]$, $n \in \mathbb{N}$, converges to F uniformly on $\overline{\mathcal{D}_{\mathbf{P}}}$. Let T be any d-tuple of commuting bounded linear operators on a Hilbert space with $\|\mathbf{P}(T)\| \leq 1$. By [3, Lemma 1], the Taylor joint spectrum of T lies in the closed domain $\overline{\mathcal{D}_{\mathbf{P}}}$ where F is analytic. By the continuity of Taylor's functional calculus [22], we have that

$$F(T) = \lim_n Q^{(n)}(T).$$

Using Lemma 3.2, we obtain that the limit

$$\lim_n \|Q^{(n)}\|_{\mathcal{A},\mathbf{P}} = \lim_n \|Q^{(n)}(T_{\max})\| = \|F(T_{\max})\|$$

exists and

$$\|F\|_{\mathcal{A},\mathbf{P}} = \sup_{T\in\mathcal{T}_{\mathbf{P}}} \|F(T)\| = \sup_{T\in\mathcal{T}_{\mathbf{P}}} \lim_n \|Q^{(n)}(T)\| \leq \lim_n \|Q^{(n)}(T_{\max})\| = \|F(T_{\max})\| < \infty.$$

□

Theorem 3.4. *Let F be a rational $\alpha\times\beta$ matrix function regular on $\overline{\mathcal{D}_{\mathbf{P}}}$ and with $\|F\|_{\mathcal{A},\mathbf{P}} < 1$. Then there exists $n = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$ and a contractive colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ of size $(\sum_{i=1}^k n_i m_i + \alpha) \times (\sum_{i=1}^k n_i \ell_i + \beta)$ such that*

$$F(z) = D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B, \qquad \mathbf{P}(z)_n = \bigoplus_{i=1}^k (\mathbf{P}_i(z) \otimes I_{n_i}).$$

Proof. Let $F = QR^{-1}$ with $\det R$ nonzero on $\overline{\mathcal{D}_{\mathbf{P}}}$ and let $\|F\|_{\mathcal{A},\mathbf{P}} < 1$. Then we have

$$R(T)^*R(T) - Q(T)^*Q(T) \geq (1 - \|F\|^2_{\mathcal{A},\mathbf{P}})R(T)^*R(T) \geq \epsilon^2 I \tag{3.1}$$

for every $T \in \mathcal{T}_{\mathbf{P}}$ with some $\epsilon > 0$. Indeed, the rational matrix function R^{-1} is regular on $\overline{\mathcal{D}_{\mathbf{P}}}$. By Lemma 3.3 $\|R^{-1}\|_{\mathcal{A},\mathbf{P}} < \infty$. Since $\|R(T)\|\|R^{-1}(T)\| \geq 1$, we obtain

$$\|R(T)\| \geq \|R^{-1}(T)\|^{-1} \geq \|R^{-1}\|^{-1}_{\mathcal{A},\mathbf{P}} > 0,$$

which yields the estimate (3.1).

By Theorem 2.3 there exist $n_0, \dots, n_k \in \mathbb{Z}_+$ and polynomials H_i with coefficients in $\mathbb{C}^{n_i m_i \times \beta}$, $i = 0, \dots, k$, (where we set $m_0 = 1$) such that by (2.1) we obtain

$$R^*(w)R(z) - Q^*(w)Q(z) = \\ H_0^*(w)H_0(z) + \sum_{i=1}^{k} H_i^*(w)\Big((I - \mathbf{P}_i^*(w)\mathbf{P}_i(z)) \otimes I_{n_i}\Big)H_i(z). \tag{3.2}$$

Denote

$$v(z) = \begin{bmatrix} (\mathbf{P}_1(z) \otimes I_{n_1})H_1(z) \\ \vdots \\ (\mathbf{P}_k(z) \otimes I_{n_k})H_k(z) \\ R(z) \end{bmatrix} \in \mathbb{C}^{(\sum_{i=1}^k \ell_i n_i + \beta)\times\beta}[z], x(z) = \begin{bmatrix} H_1(z) \\ \vdots \\ H_k(z) \\ Q(z) \end{bmatrix} \in \mathbb{C}^{(\sum_{i=1}^k m_i n_i + \alpha)\times\beta}[z].$$

Then we may rewrite (3.2) as

$$v^*(w)v(z) = H_0^*(w)H_0(z) + x^*(w)x(z). \tag{3.3}$$

Let us define

$$\mathcal{V} = \mathrm{span}\{v(z)y\colon z \in \mathbb{C}^d, y \in \mathbb{C}^\beta\}, \quad \mathcal{X} = \mathrm{span}\{x(z)y\colon z \in \mathbb{C}^d, y \in \mathbb{C}^\beta\},$$

and let $\{v(z^{(1)})y^{(1)}, \dots, v(z^{(\nu)})y^{(\nu)}\}$ be a basis for $\mathcal{V} \subseteq \mathbb{C}^{\sum_{i=1}^k \ell_i n_i + \beta}$.

Claim 1. If $v(z)y = \sum_{i=1}^\nu a_i v(z^{(i)})y^{(i)}$, then

$$x(z)y = \sum_{i=1}^{\nu} a_i x(z^{(i)})y^{(i)}.$$

Indeed, this follows from

$$0 = \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} v(z)^* \\ v(z^{(1)})^* \\ \vdots \\ v(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} v(z) & v(z^{(1)}) & \dots & v(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix} =$$

$$\begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} H_0(z)^* & x(z)^* \\ H_0(z^{(1)})^* & x(z^{(1)})^* \\ \vdots & \vdots \\ H_0(z^{(\nu)})^* & x(z^{(\nu)})^* \end{bmatrix} \times \begin{bmatrix} H_0(z) & H_0(z^{(1)}) & \cdots & H_0(z^{(\nu)}) \\ x(z) & x(z^{(1)}) & \dots & x(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix},$$

where we used (3.3). This yields

$$\begin{bmatrix} H_0(z) & H_0(z^{(1)}) & \cdots & H_0(z^{(\nu)}) \\ x(z) & x(z^{(1)}) & \dots & x(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} y \\ -a_1 y^{(1)} \\ \vdots \\ -a_\nu y^{(\nu)} \end{bmatrix} = 0,$$

and thus in particular $x(z)y = \sum_{i=1}^\nu a_i x(z^{(i)})y^{(i)}$.

We now define $S\colon \mathcal{V} \to \mathcal{X}$ via $Sv(z^{(i)})y^{(i)} = x(z^{(i)})y^{(i)}, i = 1, \ldots, \nu$. By Claim 1,

$$Sv(z)y = x(z)y \text{ for all } z \in \bigoplus_{j=1}^{k} \mathbb{C}^{\ell_j \times m_j} \text{ and } y \in \mathbb{C}^{\beta}. \tag{3.4}$$

Claim 2. S is a contraction. Indeed, let $v = \sum_{i=1}^{\nu} a_i v(z^{(i)})y^{(i)} \in \mathcal{V}$. Then $Sv = \sum_{i=1}^{\nu} a_i x(z^{(i)})y^{(i)}$, and we compute, using (3.3) in the second equality,

$$\|v\|^2 - \|Sv\|^2 =$$

$$\begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} v(z^{(1)})^* \\ \vdots \\ v(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} v(z^{(1)}) & \ldots & v(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix} -$$

$$\begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} x(z^{(1)})^* \\ \vdots \\ x(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} x(z) & x(z^{(1)}) & \ldots & x(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix} =$$

$$\begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix}^* \begin{bmatrix} H_0(z^{(1)})^* \\ \vdots \\ H_0(z^{(\nu)})^* \end{bmatrix} \begin{bmatrix} H_0(z^{(1)}) & \cdots & H_0(z^{(\nu)}) \end{bmatrix} \begin{bmatrix} a_1 y^{(1)} \\ \vdots \\ a_\nu y^{(\nu)} \end{bmatrix} \geq 0,$$

proving Claim 2.

Extending S to the contraction $S_{ext} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \mathbb{C}^{\sum_{i=1}^{k} \ell_i n_i + \beta} \to \mathbb{C}^{\sum_{i=1}^{k} m_i n_i + \alpha}$ by setting $S_{ext}|_{\mathcal{V}^\perp} = 0$, we now obtain from (3.4) that

$$A\mathbf{P}(z)_n H(z) + BR(z) = H(z), \quad C\mathbf{P}(z)_n H(z) + DR(z) = Q(z).$$

Eliminating $H(z)$, we arrive at

$$(D + C\mathbf{P}(z)_n (I - A\mathbf{P}(z)_n)^{-1} B) R(z) = Q(z),$$

yielding the desired realization for $F = QR^{-1}$. □

The following statement is a special case of Theorem 3.4.

Corollary 3.5. *Let F be a rational matrix function regular on the closed bidisk $\overline{\mathbb{D}^2}$ such that*

$$\|F\|_\infty = \sup_{(z_1,z_2)\in\mathbb{D}^2} \|F(z_1, z_2)\| < 1.$$

Then F has a finite dimensional contractive realization (1.2)*, that is, there exist $n_1, n_2 \in \mathbb{Z}_+$ such that $\mathcal{X}_i = \mathbb{C}^{n_i}$, $i = 1, 2$, and $Z_{\mathcal{X}} = z_1 I_{n_1} \oplus z_2 I_{n_2} = Z_n$.*

Proof. One can apply Theorem 3.4 after observing that on the bidisk the Agler norm and the supremum norm coincide, a result that goes back to [4]. □

4. Contractive Determinantal Representations

Let a polynomial $\mathbf{P} = \bigoplus_{i=1}^{k} \mathbf{P}_i$ and a domain $\mathcal{D}_{\mathbf{P}}$ be as in Section 3. We apply Theorem 3.4 to obtain a contractive determinantal representation for a multiple of every polynomial strongly stable on $\mathcal{D}_{\mathbf{P}}$. Please notice the analogy with the main result in [16], where a similar result is obtained in the setting of definite determinantal representation for hyperbolic polynomials

Theorem 4.1. *Let p be a polynomial in d variables $z = (z_1, \dots, z_d)$, which is strongly stable on $\mathcal{D}_{\mathbf{P}}$. Then there exists a polynomial q, nonnegative integers $n_1, \dots, n_k$, and a contractive matrix K of size $\sum_{i=1}^k m_i n_i \times \sum_{i=1}^k \ell_i n_i$ such that*

$$p(z)q(z) = \det(I - K\mathbf{P}(z)_n), \qquad \mathbf{P}(z)_n = \bigoplus_{i=1}^k (\mathbf{P}_i(z) \otimes I_{n_i}).$$

Proof. Since p has no zeros in $\overline{\mathcal{D}_{\mathbf{P}}}$, the rational function $g = 1/p$ is regular on $\overline{\mathcal{D}_{\mathbf{P}}}$. By Lemma 3.3, $\|g\|_{\mathcal{A},\mathbf{P}} < \infty$. Thus we can find a constant $c > 0$ so that $\|cg\|_{\mathcal{A},\mathbf{P}} < 1$. Applying now Theorem 3.4 to $F = cg$, we obtain a k-tuple $n = (n_1, \dots, n_k) \in \mathbb{Z}_+^k$ and a contractive colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ so that

$$cg(z) = \frac{c}{p(z)} = D + C\mathbf{P}(z)_n(I - A\mathbf{P}(z)_n)^{-1}B = \frac{\det \begin{bmatrix} I - A\mathbf{P}(z)_n & B \\ -C\mathbf{P}(z)_n & D \end{bmatrix}}{\det(I - A\mathbf{P}(z)_n)}. \tag{4.1}$$

This shows that

$$\frac{\det(I - A\mathbf{P}(z)_n)}{p(z)}$$

is a polynomial. Let $K = A$. Then K is a contraction, and

$$q(z) = \frac{\det(I - K\mathbf{P}(z)_n)}{p(z)}$$

is a polynomial. $\square$

Remark 4.2. *Since the polynomial* $\det(I - K\mathbf{P}(z)_n)$ *in Theorem 4.1 is stable on $\mathcal{D}_{\mathbf{P}}$, so is q.*

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